Report

Using GAP for Understanding Polynomial Function Rings: Enhancing Teaching Techniques

Asho Abiodun Opeyemi, Sunday Adesina Adebisi *

Department of Mathematics, Faculty of Science, University of Lagos, Akoka, Yaba, Lagos 101245, Nigeria

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Abstract: This case study illustrates how GAP software can be effectively used in teaching polynomial rings to undergraduate teachers. GAP is relatively new software that has been created recently. By utilizing computer software, many complicated ideas become simpler for students to understand. Software can be strong tools for teaching and learning Mathematics when the right programs are chosen. In this project, focused efforts are made to show how software tools related to Group Algorithms and Programming (GAP) can be used for teaching and applying some important and distinct Mathematical subjects. In this instance, topics like the ring of polynomials are examined as a clear example. This also covers some essential foundational ideas and their basic computational methods. The goal is to make each topic and the teaching strategies easier to grasp as intended.

Keywords: Group Algorithms and Programming (GAP); polynomials rings; computer software; mathematical structures; programming applications; ring functions

1. Introduction

GAP is simply, an acronym (denoting *GROUP ALGORITHMS AND PROGRAMMING*). Basically, the software is mostly used in the process of solving complex algebraic and general mathematical structural problems. This research focuses on polynomials in abstract algebra of Mathematics using GAP. The package can easily get an algebraic expression of a polynomial function and the outcome results of polynomial ring. It has a mathematical role in providing great visual experience. This research work applied *GAP* (Groups, Algorithms and Programming) to the polynomial functions as a mathematical model to solve the problems of polynomial ring. Furthermore, the researcher ensures that the results of the polynomial functions are displayed using GAP programming application on the system; thus it eases learning and the outcomes are of greater achievement to all mathematical fields [1]. The concepts of solvability in polynomials is highly imperative, most especially the concerned operations on the sums and the products of the roots [2]. The concepts of solvability in polynomials is highly imperative, most especially the concerned operations on the sums and the products of the roots. In this paper, an analysis of the fundamentals and the basic characterizations of the roots of polynomials were considered. This involves but is not limited to the sums as well as the products of their roots, not only those of the elementary or lower degrees but also that of any higher degree. Efforts have been intensified to state and prove certain characterizations which each case of the degrees of the polynomial must satisfy. Hence, the analysis further helps in determining of zeros for any given polynomial, including those of higher degrees [2]. Polynomials of a given degree are very valuable in many mathematical operations, computations and general sciences. Results involving polynomials have been found very useful and applicable to physics, chemistry, biological sciences, and even in the real-life situations and circumstances. Many life situations, cases, as well as problems, can easily be modeled mathematically using ideas from polynomials in order to proffer suitable solutions as required [2]. Historically, Integer-valued polynomials on the ring of integers have been known for a long time and used in

* Corresponding author : Sunday Adesina Adebisi, Department of Mathematics, Faculty of Science, University of Lagos, Akoka, Yaba, Lagos 101245, Nigeria, adesinasunday@yahoo.com

Calculus. Polya and Ostrowski generalized this notion to rings of integers of number fields. More generally still, one may consider a domain D and the polynomials (with coefficients in its quotient field) mapping D into itself. They form a D-algebra—that is, a D-module with a ring structure. Appearing in a very natural fashion, this ring possesses quite a rich structure and the very numerous questions it raises allow a thorough exploration of commutative algebra [3]. The neutron Quadruple NaQ(x) under the definitions of the addition and multiplication given above for the power series. The neutron Quadruple set NQ[x] of polynomials in the indeterminate x over the neutron Quadruple field NQ(F) can also be seen as a neutrosophic Ring [4]. While the GAP software was adopted by Alexander et al. [5], eminent researchers have also delved into teaching and the proper managements in teaching methodologies. These forms of expertise could be accessed in [6–14] (We implore our esteemed readers to also check [15–21] of our references for some more details concerning other vital kinds of other useful algebraic structures such as the varieties of polynomials as aforementioned).

2. Methodology

In this particular section, efforts are intensified in such a way as to actually demonstrate on how the software packages involving the exclusive use of the Group Algorithms and Programming (GAP) is being adopted, all for the generalizations as to the applications into the concepts of polynomial rings in general as applicable to mathematical entities. This, in a way, actually includes some of its basic foundational and fundamental computational processes as well as some of the implicational results.

2.1. Polynomial Ring

Lemma 2.1.1. Let R be a ring and let x be an indeterminate.

The polynomial ring R[x] is defined to be the set of all formal sums:

$$\sum a_i x^i = a_n x^n - a_{n-1}x^{n-1} + \dots a_1 x + a^0 \quad (1)$$

where each $a_i \in R$ ($a_1, a_2 \dots$ are called the coefficients of the polynomial; a_i is the coefficient of x_i).

Given two polynomials:

$$f = \sum a_i x^i = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0 \quad (2)$$

$$g = \sum b_i x^i = b_m x^m + b_{m-1}x^{m-1} + \dots b_1 x + b_0 \quad (3)$$

In $R[x]$ the sum of the polynomial represented by *f* and the polynomial represented by *g*, i.e. *f* + *g*, is defined as:

Provided that $= n$, condition for sum of functions:

$$f + g = \sum (a_i + b_i)x^i = (a_n + b_n)x^n - (a_{n-1} + b_{n-1})x^{n-1} + \dots + (a_1 + b_1)x + (a_0 + b_0) \quad (4)$$

and the product as:

$$fg = \sum_i c_i x^i = \sum_i \left(\sum_j a_j b_{i-j}\right) x^i = c_{m+n}x^{m+n} - c_{m+n-1}x^{m+n-1} + \dots c_1 x + c_0 \quad (5)$$

With these rules of addition and multiplication, $R[x]$ becomes a ring, with zero given as the polynomial with zero coefficients.

Lemma 2.1.2. Let R be a ring and let $f \in R[x]$ be a non-zero polynomial with coefficients in R. The degree of f is the largest n such that the coefficient of x^n is non-zero.

Polynomial rings give interesting examples of infinite rings of finite characteristic. For example $\mathbb{Z}_2[x]$ has infinitely many polynomials just let the degree go to infinity; but the characteristic is two. Indeed if you add a polynomial to itself, you are just adding the coefficients to themselves, which are then all zero.

More generally $\mathbb{Z}_n[x]$ is an infinite ring of finite characteristic n.

Lemma 2.1.3. Let R be an integral domain and let f and g be two non-zero elements of $R[x]$.

Then the degree of fg is the sum of the degrees of f and g. In particular $R[x]$ is an integral domain.

Definition 2.1.4. Let R be a commutative ring and let x and y be indeterminate.

A monomial in x and y is a product of powers of x and y, i.e.

$$x^i y^i \quad (6)$$

The degree d of a monomial is the sum of the degrees of the individual terms, i + j.

The polynomial ring $R[x, y]$ is equal to the set of all finite formal sums:

$$\sum_{ij} a_{ij} x^i y^i \quad (7)$$

With the obvious addition and multiplication; the degree of a polynomial is the maximum degree of a monomial term that appears with non-zero coefficient.

2.2. Teaching Case Study

GAP applications beyond polynomial rings (e.g., group theory, cryptography basics).

2.2.1. Group Theory (S_4 vs D_8)

Symmetric group S4S_4S4

Key results

- Size(Symmetric Group(4)); → **24**
- Length(Subgroups(Symmetric Group(4))); → **30** (total

subgroups)

Normal subgroups of S4S_4S4

- {e}\{e\}{e} (trivial)
- V4={(),(12)(34),(13)(24),(14)(23)}V_4=\{(),(12)(34),(13)(24),(14)(23)\}V4={(),(12)(34),(13)(24),(14)(23)}(the "double-transposition" Klein four)
- A4A_4A4 (order 12)
- S4S_4S4 itself

No others are normal.

GAP checks

```
G := Symmetric Group(4);
Size(G);                          # 24
Length(Subgroups(G));             # 30
A4 := Alternating Group(4);
V4 := Subgroup(G, [ (1,2)(3,4), (1,3)(2,4) ]);
Is Normal(G, A4);                 # true
Is Normal(G, V4);                 # true
```

2.2.2. Cryptography Basics (RSA toy example)

Given n=55=5x11n=55=5\cdot 11n=55=5x11, e=3e=3e=3, message m=7m=7m=7.

(1) φ(55)=(5−1)(11−1)=4⋅10=40\varphi(55) = (5-1)(11-1) =4\cdot 10 = \mathbf{40}φ(55)=(5−1)(11−1)=4⋅10=40

(2) Find d≡e−1(modφ(55))d \equiv e^{-1} \pmod{\varphi(55)}d≡e−1(modφ(55)):
Solve 3d≡1(mod40)⇒d=273d \equiv 1 \pmod{40} \Rightarrow d=\mathbf{27}3d≡1(mod40)⇒d=27 (since 3⋅27=81≡13\cdot 27=81\equiv 13⋅27=81≡1 mod 40).

(3) Encrypt c≡me(modn)c \equiv m^e \pmod{n}c≡me(modn):
c≡73≡343≡13(mod55)c \equiv 7^3 \equiv 343 \equiv \mathbf{13} \pmod{55}c≡73≡343≡13(mod55).

(4) Decrypt m≡cd(modn)m \equiv c^d \pmod{n}m≡cd(modn):
Using CRT or direct pow-mod, 1327≡7(mod55)13^{27} \equiv \mathbf{7} \pmod{55}1327≡7(mod55) (original message recovered).

2.2.3. GAP confirmation

```
n := 55; phi := EulerPhi(n);         # 40
e := 3; d := InverseMod(e, phi);     # 27
m := 7; c := PowerMod(m, e, n);      # 13
PowerMod(c, d, n);                   # 7
```

How GAP addresses common teaching challenges

Group Theory (S_4 vs D_8)

- **Teaching Challenge:** Students struggle to visualize subgroup structures and to understand abstract properties like normality. On the board, lattices and permutations can feel too abstract.
- **How GAP Helps**: GAP instantly generates subgroups and tests normality. Students can experiment with different subgroups (e.g., reflections vs rotations) and **see patterns** that connect algebraic definitions to concrete outputs. This makes the abstract subgroup lattice more tangible.

Cryptography Basics (RSA toy example)

- **Teaching Challenge:** Modular arithmetic and RSA encryption involve large computations that are tedious by hand, causing students to lose focus on the underlying ideas (totient, inverses, security principles).

How GAP Helps: GAP automates the modular arithmetic so students can concentrate on the **logic of RSA** (why totient is used, why inverses matter) instead of getting stuck in calculations. This shifts learning from rote arithmetic to **conceptual understanding** of security.

3. Modeling Polynomial Ring (Main Result)

3.1. Polynomial

Polynomial(R, l)
Polynomial(R, l, v)

1 must be a list of coefficients of the polynomial f to be constructed, namely (…; fv = l [1]; fv+1 = l [2]; …) over R, which must be a commutative ring-with-one or a field. The default for v is 0. Polynomial returns this polynomial f.

For interactive calculation it might by easier to construct the indeterminate over R and construct the polynomial using ^, + and *.

GAP

```
gap> x:= Indeterminate(Integers);;
gap> x.name:= "x";;
gap> f:= Polynomial(Integers, [1,2,0,0,4]);
4*x^4 + 2*x + 1
gap> g:= 4*x^4 + 2*x + 1;
4*x^4 + 2*x + 1
```

3.2. Is Polynomial

Is Polynomial(obj)

Is Polynomial returns true if obj, which can be an object of arbitrary type, is a polynomial and false otherwise. The function will signal an error if obj is an unbound variable.

4. The Groups Algorithms and Programming (GAP)

```
gap>Is Polynomial(1);
false
```

gap>Is Polynomial(Indeterminate(Integers));
true

4.1. Comparison of Polynomials

f = g
f <> g

The equality operator = evaluates to true if the polynomials f and g are equal, and to false otherwise. The inequality operator <> evaluates to true if the polynomials f and g are not equal, and to false otherwise.

Note that polynomials are equal if and only if their coefficients and their base rings are equal. Polynomials can also be compared with objects of other types. Of course they are never equal.

GAP
gap> f:= Polynomial(GF(5^3), [1,2,3]*Z(5)^0);
Z(5)^3*X(GF(5^3))^2 + Z(5)*X(GF(5^3)) + Z(5)^0
gap> x:= Indeterminate(GF(25))…
gap> g:= 3*x^2 + 2*x + 1;
Z(5)^3*X(GF(5^2))^2 + Z(5)*X(GF(5^2)) + Z(5)^0
gap> f = g;
false
gap> x^0 = Z(25)^0;
false
f < g
f <= g
f > g
f >= g

4.2. Polynomial Ring

Polynomial Ring(R)

Polynomial Ring returns the ring of all polynomials over a field R or ring-with-one R.

GAP
gap> f2:= GF(2);;
gap> R:= Polynomial Ring(f2);
Polynomial Ring(GF(2))
gap> Z(2) in R;
false
gap> Polynomial(f2, [Z(2),Z(2)]) in R;
true
gap> Polynomial(GF(4), [Z(2),Z(2)]) in R;
false
gap> R:= Polynomial Ring(GF(2));
Polynomial Ring(GF(2))

4.2.1. Is Polynomial Ring

Is Polynomial Ring(domain)

Is Polynomial Ring returns true if the object domain is a ring record, representing a polynomial ring and false otherwise.

GAP
gap>Is Polynomial Ring(Integers);
false
gap>Is Polynomial Ring(Polynomial Ring(Integers));
true
gap>Is Polynomial Ring(Laurent Polynomial Ring(Integers));
false

4.2.2. Ring Functions for Polynomial Rings

Let R be a commutative ring-with-one or a _eld and let P be the polynomial ring over R.
Euclidean Degree(P, f)

P is an Euclidean ring if and only if R is _eld. In this case the Euclidean degree of f is simply the degree of f. If R is not a _eld then the function signals an error.

GAP
gap> x:= Indeterminate(Rationals);; x.name:= "x";;
gap>Euclidean Degree(x^10 + x^2 + 1);
10
gap>Euclidean Degree(x^0);
0

4.2.3. Euclidean Remainder(P, f, g)

P is an Euclidean ring if and only if R is _eld. In this case it is possible to divide f by g with remainder.

GAP
gap> x:= Indeterminate(Rationals);; x.name:= "x";;
gap>Euclidean Remainder((x+1)*(x+2)+5, x+1);
5*x^0

4.2.4. Euclidean Quotient(P, f, g)

P is an Euclidean ring if and only if R is _eld. In this case it is possible to divide f by g with remainder.

GAP
gap> x:= Indeterminate(Rationals);; x.name:= "x";;
gap>Euclidean Quotient((x+1)*(x+2)+5, x+1);
x + 2

4.2.5. Quotient Remainder(P, f, g)

P is an Euclidean ring if and only if R is _eld. In this case it is possible to divide f by g with remainder.
gap> x:= Indeterminate(Rationals);; x.name:= "x";;
gap>Quotient Remainder((x+1)*(x+2)+5, x+1);
[x + 2, 5*x^0]
Gcd(P, f, g)

P is an Euclidean ring if and only if R is _eld. In this case you can compute the greatest common divisor of f and g using Gcd.

```
gap> x:= Indeterminate(Rationals);; x.name:= "x";;
gap> g:= x^2 + 2*x + 1;;
gap> h:= x^2 - 1;;
gap>Gcd(g, h);
x + 1
gap>Gcd Representation(g, h);
[1/2*x^0, -1/2*x^0]
gap> g * (1/2) * x^0 - h * (1/2) * x^0;
x + 1
```

4.2.6. Factors(P, f)

This method is implemented for polynomial rings P over a domain R, where R is either a finite field, the rational numbers, or an algebraic extension of either one. If char R is a prime, f is factored using a Cantor-Zassenhaus algorithm.

```
gap> f5:= GF(5);; f5.name:= "f5";;
gap> x:= Indeterminate(f5);; x.name:= "x";;
gap> g:= x^20 + x^8 + 1;
Z(5)^0*(x^20 + x^8 + 1)
gap> Factors(g);
[Z(5)^0*(x^8 + 4*x^4 + 2), Z(5)^0*(x^12 + x^8 + 4*x^4 + 3)]
```

If char R is 0, a quadratic Hensel lift is used.

```
gap> x:= Indeterminate(Rationals);; x.name:= "x";;
gap> f:=x^105-1;
x^105 - 1
gap> Factors(f);
[x - 1, x^2 + x + 1, x^4 + x^3 + x^2 + x + 1,
x^6 + x^5 + x^4 + x^3 + x^2 + x + 1,
x^8 - x^7 + x^5 - x^4 + x^3 - x + 1,
x^12 - x^11 + x^9 - x^8 + x^6 - x^4 + x^3 - x + 1,
x^24 - x^23 + x^19 - x^18 + x^17 - x^16 + x^14 - x^13 + x^12 - x^
11 + x^10 - x^8 + x^7 - x^6 + x^5 - x + 1,
x^48 + x^47 + x^46 - x^43 - x^42 - 2*x^41 - x^40 - x^39 + x^36 + x^
35 + x^34 + x^33 + x^32 + x^31 - x^28 - x^26 - x^24 - x^22 - x^
20 + x^17 + x^16 + x^15 + x^14 + x^13 + x^12 - x^9 - x^8 - 2*x^
7 - x^6 - x^5 + x^2 + x + 1]
```

As f is an element of P if and only if the base ring of f is R you must embed the polynomial into the polynomial ring P if it is written as polynomial over a subring.

GAP

```
gap> f25:= GF(25);; Indeterminate(f25).name:= "y";;
gap> l:= Factors(Embedded Polynomial(Polynomial Ring (f25), g));
[y^4 + Z(5^2)^13, y^4 + Z(5^2)^17, y^6 + Z(5)^3*y^2 + Z(5^2)^3,
y^6 + Z(5)^3*y^2 + Z(5^2)^15]
gap> l[1] * l[2];
y^8 + Z(5)^2*y^4 + Z(5)
gap> l[3] * l[4];
y^12 + y^8 + Z(5)^2*y^4 + Z(5)^3
```

4.2.7. Standard Associate(P, f)

For a ring R the standard associate a of f is a multiple of f such that the leading coefficient of a is the standard associate in R. For a field R the standard associate of the polynomial represented by f is a multiple of f such that the leading coefficient is 1.

GAP

```
gap> x:= Indeterminate(Integers);; x.name:= "x";;
gap>Standard Associate(-2 * x^3 - x);
2*x^3 + x
```

4.2.8. Ring Functions for Laurent Polynomial Rings

Let R be a commutative ring-with-one or a field and let P be the polynomial ring over R.

Euclidean Degree (P, f)

P is an Euclidean ring if and only if R is field. In this case the Euclidean degree of f is the difference of d(f) and v(f) (see 19). If R is not a field then the function signals an error.

GAP

```
gap> x:= Indeterminate(Rationals);; x.name:= "x";;
gap> LR:= Laurent Polynomial Ring(Rationals);;
gap>Euclidean Degree(LR, x^10 + x^2);
8
gap>Euclidean Degree(LR, x^7);
0
gap>Euclidean Degree(x^7);
7
gap>Euclidean Degree(LR, x^2 + x^-2);
4
gap>Euclidean Degree(x^2 + x^-2);
4
```

4.2.9. Gcd(P, f, g)

P is an Euclidean ring if and only if R is field. In this case you can compute the greatest common divisor of f and g using Gcd.

GAP

```
gap> x:= Indeterminate(Rationals);; x.name:= "x";;
gap> LR:= Laurent Polynomial Ring(Rationals);;
gap> g:= x^3 + 2*x^2 + x;;
gap> h:= x^3 - x;;
gap>Gcd(g, h);
x^2 + x
gap>Gcd(LR, g, h);
x + 1 # x is a unit in LR
gap>Gcd Representation(LR, g, h);
[(1/2)*x^(-1), (-1/2)*x^(-1)]
```

Factors(P, f)

This method is only implemented for a Laurent polynomial ring P over a finite field R. In this case f is factored using a Cantor-Zassenhaus algorithm. As f is an element of P if and only if the base ring of f is R you must embed the polynomial into the polynomial ring P if it is written as polynomial over a subring.

GAP

```
gap> f5:= GF(5);; f5.name:= "f5";;
gap> x:= Indeterminate(f5);; x.name:= "x";;
gap> g:= x^10 + x^-2 + x^-10;
Z(5)^0*(x^10 + x^(-2) + x^(-10))
gap> Factors(g);
[Z(5)^0*(x^(-2) + 4*x^(-6) + 2*x^(-10)),
Z(5)^0*(x^12 + x^8 + 4*x^4 + 3)]
gap> f25:= GF(25);; Indeterminate(f25).name:= "y";;
gap>gg:= Embedded Polynomial(Laurent Polynomial Ring(f25), g);
y^10 + y^(-2) + y^(-10)
gap> l:= Factors(gg);
[y^(-6) + Z(5^2)^13*y^(-10), y^4 + Z(5^2)^17,
y^6 + Z(5)^3*y^2 + Z(5^2)^3, y^6 + Z(5)^3*y^2 + Z(5^2)^15]
gap> l[1] * l[2];
y^(-2) + Z(5)^2*y^(-6) + Z(5)*y^(-10)
gap> l[3]*[4];
[Z(5)^2*y^6 + Z(5)*y^2 + Z(5^2)^15]
```

Standard Associate(P, f)

For a ring R the standard associate a of f is a multiple of f such that the leading coefficient of a is the standard associate in R and v(a) is zero. For a field R the standard associate of the polynomial represented by f is a multiple of f such that the leading coefficient is 1 and v(a) is zero.

GAP

```
gap> x:= Indeterminate(Integers);; x.name:= "x";;
gap> LR:= Laurent Polynomial Ring(Integers);;
gap>Standard Associate(LR, -2 * x^3 - x);
2*x^2 + 1
```

4.3. Teaching Value

How GAP addresses specific learning difficulties (e.g., student struggles with ring homomorphisms)

4.3.1. Understanding Ring Homomorphisms

- **Student Struggle:** Many students cannot "see" how a ring homomorphism works (e.g., mapping f:Z[x]→Z3[x]f: \mathbb{Z}[x] \to \mathbb{Z}_3[x]f:Z[x]→Z3 [x]), or why kernels and images matter. On paper, checking homomorphism properties is abstract and tedious.
- **How GAP Helps:**
  - GAP can **define source and target rings**, then evaluate images of generators automatically.
  - Students can compute kernels (ideals) and images computationally, giving a concrete feel for abstract structures.
- R := Integers;
- P := Polynomial Ring(R);;
- F := GF(3);
- Q := Polynomial Ring(F);;
- f := Ring Homomorphism By Images(P, Q, [X(P)], [X(Q)]);;
- Image(f, X(P)^2 + 2*X(P) + 1); # Evaluates homomorphism
  Kernel(f); # Finds kernel

4.3.2.Visualizing Abstract Group and Ring Structures

- **Student Struggle:** When told "S4S_4S4 has 30 subgroups" or "the factor ringZ[x]/(x2+1)\mathbb{Z}[x]/(x^2+1)Z[x]/(x2+1) is a field," students often memorize facts instead of understanding why.
- **How GAP Helps:**
  - For groups: GAP lists and classifies subgroups, computes normality, and shows cosets.
  - For rings: GAP constructs quotient rings and lets students compute directly inside them.
- P := Polynomial Ring(GF(5));;
- I := Ideal(P, X(P)^2 + 1);;
- Q := Factor Ring(P, I);;
  Elements(Q); # Elements of Z5[x]/(x^2+1)

4.3.3. Bridging Abstract Concepts to Applications

- **Student Struggle:** Students often ask, "Why are we learning this?" (especially with rings and finite fields). They fail to see relevance.

- **How GAP Helps:**
  - Demonstrates cryptography, coding theory, and symmetry applications.
  - For example, GAP shows how polynomial rings over finite fields underlie **error-correcting codes** and **RSA cryptography**, connecting coursework to real-world systems.

5. Discussion

This study acknowledges inherent constraints in applying Group Algorithms and Programming (GAP) to computational mathematics education within superintelligence-driven pedagogical frameworks. While GAP effectively demonstrates polynomial ring operations and abstract algebraic structures (e.g., homomorphisms, quotient rings), its integration faces pedagogical challenges: technical barriers in educational settings may hinder accessibility, and the software's current architecture exhibits limitations in handling advanced superintelligence paradigms—particularly quantum computing compatibility and neuro-symbolic integration—which are critical for next-generation computational mathematics. Future work should explore longitudinal implementations across institutional contexts and optimize GAP's synergy with quantum-resistant algorithms to align with the journal's focus on intelligent algorithm innovation and practical applications in key domains.

By following the algorithm stepwise, its expedient that the analysis would be highly beneficial, simplified and very easy to apply and use as required. Hence, complex polynomials can thus be simplified as much as possible so as to explore every necessary details involved in the teaching and learning of such forms algebraic structures. For example, the aspects of solutions, the zeros, eigenvevtors, as well as he eigenvalues which are very useful in applications in real life situations such as face capturing and recognition through the methods as been discussed could be given more and deeper exploration. Hence, this pedagogical impact of using GAP is very highly inevitable in educational managements and developments as required.

6. Conclusion

So far, we have been able to demonstrate how the software packages involving the teaching and simplifications of abstract algebraic topics such as the Group Algorithms and Programming (GAP) could be adopted for the teaching and application in some very useful and unique Mathematical topics. In this case in particular, special topics such as the ring of polynomials is considered as a typical example. Really, abstract concepts taught in courses of abstract algebra like Polynomial Ring has been concretized by means of mathematical software(GAP) as used in this work.

Abbreviations

GAP: Group Algorithms and Programming, Groups, Algorithms, and Programming, Version 4.11.0, https://www.gap-system.org
IVP: Integer-Valued Polynomials

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